Discrete Mathematics 40 (1982) 69-80
North-Holland Publishing Company

A q-ANALOG OF THE EXPONENTIAL FORMULA*

Ira M. GESSEL

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Received 1 May 1980
Revised 23 April 1981

A q-analog of functional composition for Eulerian generating functions is introduced and applied to the enumeration of permutations by inversions and distribution of left-right maxima.

1. Introduction

If $f(x)=\sum_{n=1}^{\infty} f_n(x^n/n!)$ is the exponential generating function for a class of 'labeled objects', then

$$g(x)=e^{f(x)}$$

will be (under appropriate conditions) the exponential generating function for sets of these objects. For example, if $f(x)=\sum_{n=1}^{\infty}(n-1)!\,x^n/n!$ is the exponential generating function for cyclic permutations, then $g(x)=\sum_{n=0}^{\infty} n!\,x^n/n!$ is the exponential generating function for all permutations; if $f(x)$ is the exponential generating function for connected labeled graphs, then

$$g(x)=\sum_{n=0}^{\infty} 2^{\binom{n}{2}}\frac{x^n}{n!}$$

is the exponential generating function for all labeled graphs. For various approaches to the exponential formula, see [3, 4, 10, 11].

It is well known that many properties of exponential generating functions have analogs for *Eulerian generating functions* of the form

$$\sum_{n=0}^{\infty} f_n \frac{x^n}{n!_q}$$

where $n!_q=1\cdot(1+q)\cdots(1+q+\cdots+q^{n-1})$, and f_n is a polynomial in q. Note that $n!_q$ reduces to $n!$ for $q=1$. Eulerian generating functions arise in several combinatorial applications, such as finite vector spaces [6] and partitions [1], but here we shall be concerned primarily with their use in counting permutations by inversions. (See [5, 9].)

*Research partially supported by ONR Contract #N00014-76-C-0366

0012-365X/82/0000-0000/$02.75

We introduce a q-analog of functional composition and show that q-exponentiation can be used to count permutations by inversions and 'basic components', which are related to left-right maxima. Combinatorial interpretations are obtained for Gould's q-Stirling numbers of the first kind [7] and the 'continuous q-Hermite polynomials' study by Askey and Ismail [2] and others. Finally, we count involutions by inversions, using a new property of a correspondence of Foata [4].

2. Notation

We define $(a;q)_n$ to be $\prod_{i=0}^{n-1}(1-aq^i)$, with $(a;q)_0=1$. We often write $(a)_n$ for $(a;q)_n$. Thus

$$(q)_n=(1-q)(1-q^2)\cdots(1-q^n)=(1-q)^n n!_q \quad \text{and} \quad (a)_\infty=\prod_{i=0}^{\infty}(1-aq^i).$$

The q-binomial coefficient, which is a polynomial in q, is defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix}=\frac{n!_q}{k!_q(n-k)!_q}=\frac{(q)_n}{(q)_k(q)_{n-k}}.$$

We write n_q for $\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]=1+q+\cdots+q^{n-1}$ and $\mathbf{n}$ for the set $\{1,2,\ldots,n\}$. All power series may be considered as formal, so that questions of convergence do not arise.

3. A q-analog of functional composition

The q-analog $\mathfrak{D}$ of the derivative is defined by

$$\mathfrak{D}f(x)=\frac{f(x)-f(qx)}{(1-q)x}.$$

Thus $\mathfrak{D}1=0$ and for $n>0$,

$$\mathfrak{D}\frac{x^n}{n!_q}=\frac{x^{n-1}}{(n-1)!_q}.$$

(Note that for $q=1$, $\mathfrak{D}$ reduces to the ordinary derivative.) We shall often write f' for $\mathfrak{D}f$.

We now define a q-analog of the map $f\mapsto f^k/k!$ for exponential generating functions.

Definition 3.1. Suppose that $f(0)=0$. Then for $k\geq 0$, $f^{[k]}$ is defined by $f^{[0]}=1$ and for $k>0$,

$$\mathfrak{D}f^{[k]}=f'\cdot f^{[k-1]}, \quad \text{with } f^{[k]}(0)=0. \tag{3.1}$$

Formula (3.1) is equivalent to the following recursion: let

$$f^{[k]}(x)=\sum_{n=0}^{\infty} f_{n,k}\frac{x^n}{n!_q}.$$

Then

$$f_{n+1,k}=\sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix} f_{n-j+1,1}f_{j,k-1}.$$

It is clear that $f_{n,k}=0$ for $n<k$.

As an example, take $f(x)=x^m/m!_q$. Then

$$\left(\frac{x^m}{m!_q}\right)^{[k]}=\begin{bmatrix} mk-1\\ m-1\end{bmatrix}\begin{bmatrix} m(k-1)-1\\ m-1\end{bmatrix}\cdots\begin{bmatrix} m-1\\ m-1\end{bmatrix}\frac{x^{mk}}{(mk)!_q}$$

$$=\frac{(mk)!_q}{(m!_q)^k\cdot 1\cdot(1+q^m)\cdots(1+q^m+q^{2m}+\cdots+q^{(k-1)m})}\frac{x^{mk}}{(mk)!_q}$$

Note that for $m=1$ we have $x^{[k]}=x^k/k!_q$, and for $q=1$, $(x^m/m!_q)^{[k]}$ reduces to

$$\frac{(mk)!}{m!^k k!}\frac{x^{mk}}{(mk)!}.$$

Definition 3.2. Suppose that $g(x)=\sum_{n=0}^{\infty} g_n(x^n/n!_q)$ and $f(0)=0$. Then the q-composition $g[f]$ is defined to be

$$\sum_{n=0}^{\infty} g_n f^{[n]}.$$

Note that $g[x]=g(x)$. The following is straightforward.

Proposition 3.3 (The chain rule). $\mathcal{D}g[f]=g'[f]f'$.

Unfortunately q-composition is neither associative nor distributive over multiplication, i.e., in general $(fg)[h]\neq f[h]\cdot g[h]$.

Now let $e(x)=\sum_{n=0}^{\infty} x^n/n!_q$ be the q-analog of the exponential function. Since $e'(x)=e(x)$, we have $\mathcal{D}e[f]=e[f]f'$. Equating coefficients gives a recurrence for the coefficients of $e[f]$ in terms of the coefficients of f:

Proposition 3.4. *Let $f(x)=\sum_{n=1}^{\infty} f_n(x^n/n!_q)$ and let $g(x)=\sum_{n=0}^{\infty} g_n(x^n/n!_q)=e[f]$. Then $g_0=1$ and for $n\geq 0$,*

$$g_{n+1}=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix} g_{n-k}f_{k+1}.$$

We can also express $e[f]$ as an infinite product:

Proposition 3.5. *Suppose $f(0)=0$. Then*

$$e[f]=\prod_{k=0}^{\infty}[1-(1-q)q^kxf'(q^kx)]^{-1}. \tag{3.2}$$

Proof. Let $g=e[f]$. Then $g'(x)=f'(x)g(x)$, so

$$\frac{g(x)-g(qx)}{(1-q)x}=f'(x)g(x)$$

and thus

$$g(x)=[1-(1-q)xf'(x)]^{-1}g(qx). \tag{3.3}$$

Iterating (3.3) yields (3.2). $\square$

For $f(x)=x$, Proposition 3.5 yields the well-known infinite product

$$e(x)=e[x]=\prod_{k=0}^{\infty}[1-(1-q)q^kx]^{-1}.$$

Since $e[tf(x)]=\sum_{n=0}^{\infty}t^nf^{[n]}$, we have an alternative characterization of $f^{[n]}$ as the coefficient of t^n in

$$\prod_{k=0}^{\infty}[1-(1-q)q^kxf'(q^kx)t]^{-1}.$$

4. Permutations

By a *permutation* of a set A of positive integers we mean a linear arrangement $a_1a_2\cdots a_n$ of the elements of A. The *length* of $a_1a_2\cdots a_n$ is n. A permutation is *basic* if it begins with its greatest element. (By convention the 'empty permutation' of length zero is not basic.) We denote by S_n and B_n the sets of all permutations and of basic permutations of $\mathbf{n}$. (Thus $|S_n|=n!$ for all n and $|B_n|=(n-1)!$ for $n\geq 1$, with $|B_0|=0$.) A *left-right maximum* of a permutation $a_1a_2\cdots a_n$ is an a_j such that $i<j$ implies $a_i<a_j$. For any nonempty permutation σ we write $L(\sigma)$ for the first element of σ. The following is straightforward.

Lemma 4.1. *Suppose the permutation $\pi=a_1a_2\cdots a_n$ has the factorization $\pi=\beta_1\beta_2\cdots\beta_k$, where the β_i are nonempty permutations. Then the following are equivalent*:

(i) *Each β_s is basic and $L(\beta_1)<L(\beta_2)<\cdots<L(\beta_k)$.*

(ii) *$a_j=L(\beta_s)$ for some s if and only if a_j is a left-right maximum.*

It follows from the lemma that every permutation π has a unique factorization $\beta_1\beta_2\cdots\beta_k$ satisfying (i) which we call the *basic decomposition* of π, and we call the β_i the *basic components* of π. We note that any set $\{\beta_1,\ldots,\beta_k\}$ of basic permutations with no elements in common can be ordered in exactly one way to form the basic decomposition of some permutation. Thus we have a bijection between permutations and sets of disjoint basic permutations.

We call a permutation *reduced* if it is in S_n for some $n \geq 0$. To any permutation $\pi = a_1 a_2 \cdots a_n$ we may associate a reduced permutation, red(π), by replacing in π, for each $i = 1, 2, \ldots, n$, the ith smallest element of $\{a_1, a_2, \ldots, a_n\}$ by i. Thus red(7926) = 3412. The *content* of the permutation $\pi = a_1 a_2 \cdots a_n$ is $\text{con}(\pi) = \{a_1, a_2, \ldots, a_n\}$. We note that a permutation is determined by its reduction and its content.

A function ω defined on permutations (with values in some commutative algebra over the rationals) is *multiplicative* if for all permutations π:

(i) $\omega(\pi) = \omega(\text{red}(\pi))$.

(ii) If $\beta_1 \beta_2 \cdots \beta_k$ is the basic decomposition of π, then

$$\omega(\pi) = \omega(\beta_1)\omega(\beta_2) \cdots \omega(\beta_k).$$

Thus a multiplicative function is determined by its values on reduced basic permutations, and these may be chosen arbitrarily. (We note that (ii) implies $\omega(\emptyset) = 1$.)

5. Inversions of permutations

If V is a subset of $\mathbf{n}$ we denote by $I_n(V)$ the number of pairs (v, w) with $v \in V$, $w \in \mathbf{n} - V$, and $v > w$.

Lemma 5.1. *Let*

$$Q(n, k) = \sum_V q^{I_n(V)}$$

where the sum is over all $V \subseteq \mathbf{n}$ *with* $|V| = n - k$. *Then* $Q(n, k) = \left[{n \atop k}\right]$.

Proof. It is clear that $Q(n, n) = Q(n, 0) = 1$ for all $n \geq 0$. Then by considering the two cases $n \in V$ and $n \notin V$ we find the recurrence

$$Q(n, k) = q^k Q(n-1, k) + Q(n-1, k-1),$$

for $0 < k < n$. Since $\left[{n \atop k}\right]$ satisfies the same recurrence and boundary conditions, $Q(n, k) = \left[{n \atop k}\right]$. □

An *inversion* of the permutation $\pi = a_1 a_2 \cdots a_n$ is a pair (i, j) with $i < j$ and $a_i > a_j$. We write $I(\pi)$ for the number of inversions of π. Note that $I(\pi) = I(\text{red}(\pi))$.

Theorem 5.2. *Let* ω *be a multiplicative function on permutations. Let* $g_n = \sum_{\pi \in S_n} \omega(\pi) q^{I(\pi)}$ *and let* $f_n = \sum_{\beta \in B_n} \omega(\beta) q^{I(\beta)}$. *Then*

$$\sum_{n=0}^{\infty} g_n \frac{x^n}{n!_q} = e\left[\sum_{n=1}^{\infty} f_n \frac{x^n}{n!_q}\right].$$

Proof. In view of Proposition 3.4, we need only prove

$$g_{n+1}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}g_{n-k}f_{k+1}. \tag{5.1}$$

We shall prove (5.1) by showing that $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]g_{n-k}f_{k+1}$ counts those permutations counted by g_{n+1} whose last basic component has length $k+1$. Such a permutation may be factored as $\pi=\sigma\beta$ where σ is of length $n-k$, β is of length $k+1$, and the disjoint union of $\mathrm{con}(\sigma)$ and $\mathrm{con}(\beta)$ is $\mathbf{n+1}$. The condition that β is the last basic component of π is equivalent to the condition that β is basic and $\mathrm{con}(\beta)$ contains $n+1$. Thus to determine π we choose $V=\mathrm{con}(\sigma)$ as an arbitrary $(n-k)$-subset of $\mathbf{n}$ and choose $\mathrm{red}(\sigma)\in S_{n-k}$ and $\mathrm{red}(\beta)\in B_{k+1}$. It is easily seen that $I(\pi)=I(\sigma)+I(\beta)+I_n(V)$. Thus the contribution to g_{n+1} of these π is

$$\sum_{V}\sum_{\sigma\in S_{n-k}}\sum_{\beta\in B_{k+1}}\omega(\sigma)\omega(\beta)q^{I(\sigma)+I(\beta)+I_n(V)}$$
$$=\left[\sum_{V}q^{I_n(V)}\right]\left[\sum_{\sigma\in S_{n-k}}\omega(\sigma)q^{I(\sigma)}\right]\left[\sum_{\beta\in B_{k+1}}\omega(\beta)q^{I(\beta)}\right]$$
$$=\begin{bmatrix}n\\k\end{bmatrix}g_{n-k}f_{k+1},\quad\text{by Lemma 5.1.}\quad\Box$$

Corollary 5.3. *Let $t_1,t_2,\ldots$ be arbitrary, and set $T(x)=\sum_{n=0}^{\infty}t_{n+1}x^n$. Define the multiplicative function ω by*

$$\omega(\pi)=t_1^{b_1}t_2^{b_2}\cdots$$

where π has b_i basic components of length i. Let

$$g_n=\sum_{\pi\in S_n}\omega(\pi)q^{I(\pi)}$$

and let

$$g(x)=\sum_{n=0}^{\infty}g_n\frac{x^n}{n!_q}.$$

Then

$$g(x)=\prod_{k=0}^{\infty}[1-(1-q)q^kxT(q^{k+1}x)]^{-1}. \tag{5.2}$$

Proof. Let $f_n=\sum_{\beta\in B_n}\omega(\beta)q^{I(\beta)}=t_n\sum_{\beta\in B_n}q^{I(\beta)}$. Every β in B_n is obtained by inserting n at the beginning of an element of S_{n-1}; thus,

$$\sum_{\beta\in B_n}q^{I(\beta)}=q^{n-1}\sum_{\pi\in S_{n-1}}q^{I(\pi)}=q^{n-1}(n-1)!_q,$$

by a well-known result of Rodrigues [8], easily proved by induction. Thus,

$$f(x)=\sum_{n=1}^{\infty}f_n\frac{x^n}{n!_q}=\sum_{n=1}^{\infty}t_nq^{n-1}(n-1)!_q\frac{x^n}{n!_q},$$

so

$$f'(x)=\sum_{n=0}^{\infty} t_{n+1}q^n x^n = T(qx).$$

Then (5.2) follows from Theorem 5.2 and Proposition 3.5. □

6. Examples

We first look at two trivial cases of Theorem 5.2. If we take $t_i=1$ for all i, then $T(x)=(1-x)^{-1}$ and

$$\begin{aligned} g(x) &= \prod_{k=0}^{\infty} [1-(1-q)q^k x(1-q^{k+1}x)^{-1}]^{-1} \\ &= \prod_{k=0}^{\infty} \frac{1-q^{k+1}x}{1-q^k x} \\ &= \frac{1}{1-x} = \sum_{n=0}^{\infty} n!_q \frac{x^n}{n!_q}. \end{aligned}$$

If we take $t_1=1$ and $t_i=0$ for $i>1$, then $T(x)=1$ and

$$g(x)=\prod_{k=0}^{\infty}[1-(1-q)q^k x]^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{n!_q}.$$

A more interesting example is that in which $t_i=t$ for all i. In the case $q=1$ we have

$$\begin{aligned} g(x) &= \exp\left(t\sum_{n=1}^{\infty}\frac{x^n}{n}\right) = (1-x)^{-t} \\ &= \sum_{n,k=0}^{\infty} c(n,k)t^k \frac{x^n}{n!} \end{aligned}$$

where $c(n,k)=|s(n,k)|$ is the unsigned Stirling number of the first kind.

For general q, we have $T(x)=t(1-x)^{-1}$, and thus

$$\begin{aligned} g(x) &= \prod_{k=0}^{\infty}[1-(1-q)q^k xt(1-q^{k+1}x)^{-1}]^{-1} \\ &= \prod_{k=0}^{\infty} \frac{1-q^{k+1}x}{1-[q+(1-q)t]q^k x} \\ &= \frac{(qx)_\infty}{([q+(1-q)t]x)_\infty}. \end{aligned} \tag{6.1}$$

We can expand this product with the q-binomial theorem [1, p. 17]:

$$\frac{(ax)_\infty}{(x)_\infty} = \sum_{n=0}^{\infty} (a)_n \frac{x^n}{(q)_n},$$

which with βx for x and $\alpha\beta^{-1}$ for a, gives

$$\frac{(\alpha x)_\infty}{(\beta x)_\infty}=\sum_{n=0}^{\infty}\left[\prod_{i=0}^{n-1}(\beta-\alpha q^i)\right]\frac{x^n}{(q)_n},$$

where as usual the empty product is one. Then (6.1) becomes

$$\begin{aligned} g(x)&=\sum_{n=0}^{\infty}\prod_{i=0}^{n-1}[(1-q)t+q-q^{i+1}]\frac{x^n}{(q)_n}\\ &=\sum_{n=0}^{\infty}\prod_{i=0}^{n-1}[t+q(1+q+\cdots+q^{i+1})]\frac{x^n}{n!_q}\\ &=1+\sum_{n=1}^{\infty}t(t+q\cdot 1_q)(t+q\cdot 2_q)\cdots(t+q(n-1)_q)\frac{x^n}{n!_q}. \end{aligned}\tag{6.2}$$

It should be noted that a direct combinatorial proof of (6.2) is not difficult. It follows from (6.2) that the coefficients of $g(x)$ are essentially the same q-Stirling numbers as those studied by Gould [7].

With the help of the formula [1, p. 36]

$$\prod_{i=0}^{n-1}(\alpha+\beta q^i)=\sum_{j=0}^{n}q^{\binom{j}{2}}\begin{bmatrix}n\\ j\end{bmatrix}\alpha^{n-j}\beta^j,$$

one can obtain the explicit formula

$$c_q(n,k)=\left(\frac{q}{1-q}\right)^{n-k}\sum_{j=0}^{n}(-1)^j\binom{n-j}{k}q^{\binom{j}{2}}\begin{bmatrix}n\\ j\end{bmatrix}.\tag{6.3}$$

(See Gould [7].)

It is remarkable that there seems to be no formula for the ($q=1$) Stirling numbers of the first kind as simple as (6.3).

As a generalization, we may count permutations in which every basic component has length divisible by some positive integer r, according to the number of basic components. (The last example is the case $r=1$.) Here we have $T(x)=tx^{r-1}/(1-x^r)$ and a straightforward computation yields

$$\begin{aligned} g(x)&=\frac{(q^rx^r;q^r)_\infty}{([q+(1-q)t]q^{r-1}x^r;q^r)_\infty}\\ &=\sum_{n=0}^{\infty}q^{(r-1)n}\frac{(nr)!_q}{r_q(2r)_q\cdots(nr)_q}\prod_{i=0}^{n-1}[t+q(ri)_q]\frac{x^{nr}}{(nr)!_q}. \end{aligned}$$

Next, let us consider the case where all basic components have length one or two. Then we may set $t_1=t$, $t_2=1$, and $t_i=0$ for $i>2$. (Letting t_2 be an indeterminate would give us no additional information.) Then $T(x)=t+x$ and we have

$$g(x)=e\left[tx+q\frac{x^2}{2!_q}\right].$$

Proposition 3.4 gives the recurrence

$$g_{n+1} = tg_n + qn_q g_{n-1}$$

from which the first few values of g_n are easily computed:

$$g_0 = 1,$$
$$g_1 = t,$$
$$g_2 = t^2 + q,$$
$$g_3 = t^3 + (2q + q^2)t,$$
$$g_4 = t^4 + (3q + 2q^2 + q^3)t^2 + q^2 + q^3 + q^4.$$

The infinite product for $g(x)$ is

$$g(x) = \prod_{k=0}^{\infty} [1 - (1-q)q^k x(t + q^{k+1}x)]^{-1}. \tag{6.4}$$

To find a formula for the coefficients of $g(x)$ we introduce the 'continuous q-Hermite polynomials' $H_n(u \mid q)$ defined by

$$\prod_{k=0}^{\infty} (1 - 2uzq^k + z^2q^{2k})^{-1} = \sum_{n=0}^{\infty} H_n(u \mid q) \frac{z^n}{(q)_n} \tag{6.5}$$

These polynomials have been studied by Askey and Ismail [2] and others. We find a formula for their coefficients by setting $u = \cos\theta$, $\alpha = e^{i\theta}$, and $\beta = e^{-i\theta}$. Then $1 - 2uzq^k + z^2q^{2k} = (1 - \alpha zq^k)(1 - \beta zq^k)$ so

$$\prod_{k=0}^{\infty} (1 - 2uzq^k + z^2q^{2k})^{-1} = (\alpha z)_\infty^{-1}(\beta z)_\infty^{-1}$$

$$= \left[\sum_{n=0}^{\infty} \alpha^n \frac{z^n}{(q)_n}\right]\left[\sum_{n=0}^{\infty} \beta^n \frac{z^n}{(q)_n}\right].$$

Equating coefficients of $z^n/(q)_n$ and using the well-known formula

$$\cos r\theta = \sum_{m=0}^{[r/2]} (-1)^m 2^{r-2m-1} \frac{r}{r-m} \binom{r-m}{m} \cos^{r-2m}\theta$$

for $r > 0$, we obtain

$$H_n(u \mid q) = \sum_{j=0}^{[(n-1)/2]} (2u)^{n-2j} \sum_{k=0}^{j} (-1)^{j-k} \frac{n-2k}{n-k-j} \binom{n-k-j}{n-2j} \begin{bmatrix} n \\ k \end{bmatrix} + E_n \tag{6.6}$$

where

$$E_n = \begin{cases} \begin{bmatrix} n \\ \frac{1}{2}n \end{bmatrix}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

It will be convenient to consider the polynomials $\bar{H}_n(u \mid q) = i^n H_n(-iu \mid q)$, where

$i=\sqrt{-1}$. Then (6.5) and (6.6) lead to

$$\prod_{k=0}^{\infty}(1-2uzq^{k}-z^{2}q^{2k})^{-1}=\sum_{n=0}^{\infty}\bar{H}_n(u\mid q)\frac{z^n}{(q)_n} \tag{6.7}$$

and

$$\bar{H}_n(u\mid q)=\sum_{j=0}^{[(n-1)/2]}(2u)^{n-2j}\sum_{k=0}^{j}(-1)^k\frac{n-2k}{n-k-j}\binom{n-k-j}{n-2j}\begin{bmatrix}n\\k\end{bmatrix}+E_n. \tag{6.8}$$

Now in (6.4), set $z^2=(1-q)qx^2$, so $x=[(1-q)q]^{-1/2}z$. Then

$$\begin{aligned}g(x)&=\sum_{n=0}^{\infty}\bar{H}_n\left(\frac{1}{2}\left(\frac{1-q}{q}\right)^{\frac12}t\mid q\right)\frac{z^n}{(q)_n}\\&=\sum_{n=0}^{\infty}[q/(1-q)]^{n/2}\bar{H}_n\left(\frac{1}{2}\left(\frac{1-q}{q}\right)^{\frac12}t\mid q\right)\frac{x^n}{n!_q}.\end{aligned}$$

Thus

$$\begin{aligned}g_n&=[q/(1-q)]^{n/2}\bar{H}_n\left(\frac{1}{2}\left(\frac{1-q}{q}\right)^{\frac12}t\mid q\right)\\&=\sum_{j=0}^{[(n-1)/2]}t^{n-2j}\left(\frac{q}{1-q}\right)^{j}\sum_{k=0}^{j}(-1)^k\frac{n-2k}{n-k-j}\binom{n-k-j}{n-2j}\begin{bmatrix}n\\k\end{bmatrix}\\&\qquad+\left(\frac{q}{1-q}\right)^{n/2}E_n.\end{aligned} \tag{6.9}$$

Then if $g_n=\sum_j b_{n,j}t^{n-2j}$ we have

$$b_{n,0}=1,$$

$$b_{n,1}=\frac{q}{1-q}(n-n_q)=(n-1)q+(n-2)q^2+\cdots+q^{n-1},$$

$$b_{n,2}=\left(\frac{q}{1-q}\right)^2\left\{\frac{n(n-3)}{2}-(n-2)n_q+\begin{bmatrix}n\\2\end{bmatrix}\right\},$$

and so on.

7. Inversions and cycle structure

It is well known that the number of permutations in S_n with k left-right maxima is the same as the number of permutations in S_n with k cycles. (The number is the unsigned Stirling number of the first kind $c(n,k)$.) Foata [4] has constructed a bijection $\Psi: S_n\to S_n$ which takes a permutation with α_i basic components of length i (for each i) to one with α_i cycles of length i: to get the cycle representation of $\Psi(\pi)$, we simply enclose each basic component of π in a pair of parentheses. Thus for $\pi=1\,4\,2\,3\,7\,5\,6$, we have $\Psi(\pi)=(1)(4\,2\,3)(7\,5\,6)$ in cycle notation, which in linear notation is $1\,3\,4\,2\,6\,7\,5$. To find $\Psi^{-1}(\pi)$, we write π in cycle notation, with the greatest element of each cycle first, and with the cycles arranged in increasing order of first element. Then we remove the parentheses.

Unfortunately, Ψ does not preserve inversions, and the problem of counting permutations by inversions and cycle structure remains open. However, if π has only basic components of lengths one and two, so that $\Psi(\pi)$ is an involution, then Ψ transforms the inversion number in a very simple way:

Theorem 7.1. *Suppose π has b_i basic components of length i for each i, where $b_i = 0$ for $i > 2$. Then $I(\Psi(\pi)) = 2I(\pi) - b_2$.*

Proof. We proceed by induction on the length of π. The theorem is trivially true for lengths zero and one. Now let π be of length $n \geqslant 2$ and assume the truth of the theorem for all shorter lengths. Let π' be obtained from π by removing the last basic component, and let b_2' be the number of basic components of π' of length two. If the last basic component of π has length one, then $\Psi(\pi)$ is $\Psi(\pi')$ with n adjoined at the end, so $I(\pi) = I(\pi')$, $I(\Psi(\pi)) = I(\Psi(\pi'))$, and $b_2 = b_2'$. Thus $I(\Psi(\pi)) - 2I(\pi) + b_2 = I(\Psi(\pi')) - 2I(\pi') + b_2' = 0$.

To deal with the case in which the last basic component of π has length two, we first observe that Foata's correspondence Ψ can be extended in the obvious way to permutations that are not reduced: $\Psi(\sigma)$ is defined by $\operatorname{con}(\Psi(\sigma)) = \operatorname{con}(\sigma)$ and $\operatorname{red}(\Psi(\sigma)) = \Psi(\operatorname{red}(\sigma))$.

If the last basic component of π has length two, then it must be nk for some k. Then $I(\pi) = I(\pi') + n - k$. If $\Psi(\pi') = a_1 a_2 \cdots a_{n-2}$, then $\Psi(\pi)$ is obtained from it by inserting n between a_{k-1} and a_k (or at the beginning, if $k = 1$), and inserting k at the end. It is then easily seen that $I(\Psi(\pi)) = I(\Psi(\pi')) + 2n - 2k - 1$. Since $b_2 = b_2' + 1$, we have

$$\begin{aligned} I(\Psi(\pi)) - 2I(\pi) + b_2 &= I(\Psi(\pi')) + 2n - 2k - 1 - 2[I(\pi') + n - k] + b_2' + 1 \\ &= I(\Psi(\pi')) + 2I(\pi') + b_2' = 0. \quad \square \end{aligned}$$

It follows that if $g_n = g_n(t \mid q)$ is given by (6.9), then the number of involutions of **n** with r fixed points and I inversions is the coefficient of $t^r q^I$ in $q^{-n/2} g_n(tq^{\frac{1}{2}} \mid q^2)$.

References

[1] G.E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Applications, Vol. 2 (Addison-Wesley, Reading, MA, 1976).

[2] R. Askey and M.E.-H. Ismail, A generalization of ultraspherical polynomials, to appear in Studies in Analysis, A collection of articles dedicated to the memory of Paul Turan, Budapest.

[3] E.A. Bender and J.R. Goldman, Enumerative uses of generating functions, Indiana Univ. Math. J. 20 (1971) 753–765.

[4] D. Foata, La série génératrice exponentielle dans les problèmes d'énumération (Les Presses de l'Université de Montréal, 1974).

[5] I.M. Gessel, Generating functions and enumeration of sequences, Ph.D. Thesis, Massachusetts Institute of Technology, 1977.

[6] J. Goldman and G.-C. Rota, On the foundations of combinatorial theory IV: Finite vector spaces and Eulerian generating functions, Studies in Appl. Math. 49 (1970) 239–258.

[7] H.W. Gould, q-Stirling numbers, Duke Math. J. 28 (1961) 281–289.
[8] O. Rodrigues, Note sur les inversions, ou derangements produit dans les permutations, J. Mathematiques 4 (1839) 236–240.
[9] R.P. Stanley, Binomial posets, Möbius inversion, and permutation enumeration, J. Combin. Theory (A) 20 (1976) 336–356.
[10] R.P. Stanley, Generating functions, in: G.-C. Rota, ed., MAA Studies in Combinatorics (Math. Assoc. of America, 1978) 100–141.
[11] R.P. Stanley, Exponential structures, Studies in Appl. Math. 59 (1978) 73–82.